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An Implication in Orthologic[1]

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We involve a certain propositional logic based on an ortholattice. We characterize the implication reduct of such a logic and show that its algebraic counterpart is the so-called orthosemilattice. Properties of congruences and congruence kernels of these algebras are described.

KEY WORDS: ortholattice; orthosemilattice; implication orthoalagebra; congruence; congruence kernel.
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By an *ortholattice* is meant an algebra $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L; \vee, \wedge)$ is a lattice with the least element 0 and the greatest one 1 and $^{\perp}$ denotes a complementation which is involutory, i.e. $x^{\perp\perp} = x$ for each $x \in L$ and $x \leq y$ in L implies $y^{\perp} \leq x^{\perp}$ (which is equivalent to De Morgan laws: $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$ and $(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}$). Of course, every Boolean algebra and every orthomodular lattice are ortholattices. However, a Boolean algebra serves as algebraic counterpart of classical propositional logic where $\vee$ or $\wedge$ stand for disjunction or conjuction, respectively, and the complement x' of x as a negation. Then the logical connective implication can be derived by

$$x \Rightarrow y = x' \vee y.$$

On the other hand, an orthomodular lattice can analogously serve as an algebraic counterpart of the so-called logic of quantum mechanics, shortly the so-called orthomodular logic, see (Chajda *et al.*, 2001). In such a logic, the connective implication is expressed by means of $\vee$, $\wedge$ and complementation as follows:

$$x \Rightarrow y = (x^{\perp} \wedge y^{\perp}) \vee y.$$

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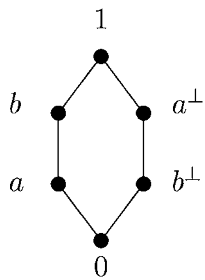

Fig. 1.

Unfortunately, in ortholattices the analogy does not work. If we consider an ortholattice visualized in Fig. 1, then for $x \Rightarrow y := x^{\perp} \vee y$ we have

$$a \Rightarrow b = 1 \quad \text{and} \quad b \Rightarrow a = 1$$

which contradicts to the accepted logical rules.

Hence, we improve the object of our considerations as follows:

Definition 1. An ortholattice $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ is called a *strong ortholattice* if for each $p \in L$ the interval $[p, 1]$ is also an ortholattice with respect to induced order, i.e. $([p, 1]; \vee, \wedge, {}^{\perp}_{p}, p, 1)$ is an ortholattice where for $a, b \in [p, 1]$ the operations $a \vee b$, $a \wedge b$ coincide with those of $\mathcal{L}$ and there exists an orthocomplement $a_p^{\perp}$ in $[p, 1]$ for each $a \in [p, 1]$.

Example. A strong ortholattice which is neither modular (since $\{0, e, d, b^{\perp}, 1\}$ is a sublattice isomorphic to N_5) nor orthomodular (since $a \leq c^{\perp}$ but $a \vee (a^{\perp} \wedge c^{\perp}) = a \neq c^{\perp}$) is depicted in Fig. 2.

For our purposes, a weaker structure is convenient, i.e. we will consider only order-filters in a strong ortholattice which will be called orthosemilattice, precisely:

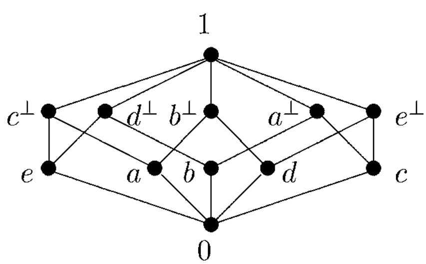

Fig. 2.

Definition 2. Let $\mathcal{S} = (S, \vee)$ be a semilattice with the greatest element 1 where for each $p \in S$ the interval $[p, 1]$ is an ortholattice with respect to induced order; denote by $a_p^{\perp}$ the orthocomplement of $a \in [p, 1]$ in $[p, 1]$ and for $([p, 1]; \vee, \wedge, {}_p^{\perp}, p, 1)$ we have

$$a \wedge b = (a_p^{\perp} \vee b_p^{\perp})_p^{\perp},$$

where $\vee$ coincides with that of $\mathcal{S}$. Then $\mathcal{S}$ is called an *orthosemilattice.*

As it was already mentioned, each order-filter in a strong ortholattice is an orthosemilattice. Every orthosemilattice is a set-theoretical union of strong ortholattices where the operations $\vee$ and $\wedge$ coincide on the overlapping parts.

Theorem 1. *Let $\mathcal{S} = (S; \vee)$ be an orthosemilattice. Define the operation "$\bullet$" as follows:*

$$x \bullet y := (x \vee y)_y^{\perp}.$$

Then

(a) $a \bullet 1 = 1, a \bullet a = 1, 1 \bullet a = a$
(b) $(a \bullet b) \bullet b = (b \bullet a) \bullet a$
(b) $(((a \bullet b) \bullet b) \bullet p) \bullet (a \bullet p) = 1$
(d) $(((a \bullet p) \bullet p) \bullet p) \bullet ((a \bullet p) \bullet p) = (a \bullet p) \bullet p.$

Proof:

(a) Clearly $a \bullet 1 = (a \vee 1)_1^{\perp} = 1$
$a \bullet a = (a \vee a)_a^{\perp} = a_a^{\perp} = 1$
$1 \bullet a = (1 \vee a)_a^{\perp} = 1_a^{\perp} = a.$

(b) Since $a \vee b \geq b$ then $a \vee b \in [b, 1]$, also $(a \vee b)_b^{\perp} \in [b, 1]$ and hence $(a \vee b)_b^{\perp} \geq b$. Then $(a \bullet b) \bullet b = ((a \vee b)_b^{\perp} \vee b)_b^{\perp} = ((a \vee b)_b^{\perp})_b^{\perp} = a \vee b$. Analogously, $(b \bullet a) \bullet a = b \vee a = a \vee b = (a \bullet b) \bullet b$.

(c) Since $a \vee b = (a \bullet b) \bullet b$, (c) can be rewritten as $((a \vee b) \bullet p) \bullet (a \bullet p) = 1$. It is easily seen that (c) is equivalent to

$$a \leq b \ \Rightarrow \ b \bullet p \leq a \bullet p.$$

Suppose $a \leq b$. Then $p \leq a \vee p \leq b \vee p$. Since the orthocomplementation in $[p, 1]$ converses the order, we obtain $a \bullet p = (a \vee p)_p^{\perp} \geq (b \vee p)_p^{\perp} = b \bullet p$.

(d) Similarly, (d) is equivalent to the condition
(d') $$p \leq a \ \Rightarrow \ ((a \bullet p) \bullet a) \bullet a = 1.$$

Indeed, let (d) hold and $p \leq a$. Then $(a \bullet p) \bullet a = [(a \vee p) \bullet p] \bullet (a \vee p) = (((a \bullet p) \bullet p) \bullet p) \bullet ((a \bullet p) \bullet p) = a$, whence

$$((a \bullet p) \bullet a) \bullet a = 1.$$

Conversely, assume that $p \leq a$ and $((a \bullet p) \bullet a) \bullet a = 1$. Then $(a \bullet p) \bullet a \leq a$ and since $a \leq (a \bullet p) \bullet a$, we have $(a \bullet p) \bullet a = a$. Finally, replacing a by $a \vee p$ in the previous equality, we obtain

$$[(a \vee p) \bullet p] \bullet (a \vee p) = a \vee p,$$

hence (d) holds.

If $p \leq a$, i.e. $a \in [p, 1]$, then $((a \bullet p) \bullet a) \bullet a = (a \bullet p) \vee a = (a \vee p)_p^{\perp} \vee a = a_p^{\perp} \vee a = 1$. □

Definition 3. An algebra $\mathcal{A} = (A; \bullet, 1)$ of type $(2, 0)$ satisfying the identities (a),(b),(c),(d) of Theorem 1 will be called an *implication orthoalgebra*.

Remark. The name implication orthoalgebra is motivated by the fact that the operation "•" can be considered as the logical connective implication. For the sake of brevity, we shall write $x \bullet y$ instead of $x \Rightarrow y$, analogously as in Abbott (1967) where this operation stands for the implication in a classical logic.

Theorem 2. Let $\mathcal{A} = (A; \bullet, 1)$ *be an implication orthoalgebra. For* $x, y \in A$ *define*

$$x \leq y \text{ iff } x \bullet y = 1$$

$$x \vee y := (x \bullet y) \bullet y$$

and for $p \in A$ *and* $a, b \in [p, 1]$ *define*

$$a \wedge b := (((a \bullet p) \bullet (b \bullet p)) \bullet (b \bullet p)) \bullet p.$$

Then $\leq$ *is an order on A with the greatest element 1, and* $x \vee y = sup(x, y)$ *with respect to* $\leq$, *i.e.* $(A; \vee)$ *is a* $\vee$*-semilattice with the greatest element* 1. *For each* $p \in A$ *the interval* $[p, 1]$ *is a lattice with respect to* $\vee, \wedge$ *as defined above and* $a \bullet p$ *is an orthocomplement of* $a \in [p, 1]$. *Hence,* $(A; \vee)$ *is an orthosemilattice.*

Proof: By (a), $\leq$ is reflexive. Suppose $a \leq b$ and $b \leq a$. Then $a \bullet b = 1$ and $b \bullet a = 1$ and we derive by (a) and (b) also $a = 1 \bullet a = (b \bullet a) \bullet a = (a \bullet b) \bullet b = 1 \bullet b = b$, thus $\leq$ is antisymmetric.

Let $a \leq b$ and $b \leq c$. By (c) we have $1 = b \bullet c \leq a \bullet c$ which yields $a \bullet c = 1$, i.e. $a \leq c$. Thus $\leq$ is also transitive, i.e. it is an order on A.

Since $a \bullet 1 = 1$ by (a), 1 is the greatest element w.r.t. $\leq$.

Put now $a \vee b := (a \bullet b) \bullet b$. If $a \leq b$ then $a \bullet b = 1$ and hence $a \vee b = (a \bullet b) \bullet b = 1 \bullet b = b$, i.e.

$$(*) \quad a \leq b \Rightarrow a \vee b = b.$$

Further, $a \vee b = (a \bullet b) \bullet b \geq 1 \bullet b = b$ (by (c)) and $a \vee b = b \vee a = (b \bullet a) \bullet a \geq 1 \bullet a = a$ thus $a \leq a \vee b, b \leq a \vee b$. Suppose $a \leq c, b \leq c$. Then, by (c),

$a \vee b = (a \bullet b) \bullet b \leq (c \bullet b) \bullet b = c \vee b$. By $(*)$ and (b) we have $c \vee b = c$, i.e. $a \vee b \leq c$. We have shown that $a \vee b = sup(a, b)$ with respect to $\leq$.
Let $p \in A$ and $a, b \in [p, 1]$. By (c) we obtain

$$a \leq b \text{ implies } b \bullet p \leq a \bullet p$$

$$a = a \vee p = (a \bullet p) \bullet p$$

thus the mapping $a \longmapsto a \bullet p$ for $a \in [p, 1]$ is an involutory antiautomorphism of $([p, 1], \leq)$ which implies De Morgan laws

$$(x \vee y) \bullet p = (x \bullet p) \wedge (y \bullet p), (x \wedge y) \bullet p = (x \bullet p) \vee (y \bullet p)$$

where $x \wedge y := ((x \bullet p) \vee (y \bullet p)) \bullet p$.

This implies that $x \wedge y = inf(x, y)$ in $[p, 1]$ w.r.t $\leq$ (restricted to the interval $[p, 1]$).

Moreover, for $a \in [p, 1]$ denote by $a_p^\perp$ the element $a \bullet p$. Then $p \leq a$ implies

$$a = a \vee p = (a \bullet p) \bullet p = (a_p^\perp)_p^\perp$$

and, by (d'),

$$a_p^\perp \vee a = ((a \bullet p) \bullet a) \bullet a = 1.$$

Further,

$$a \wedge a_p^\perp = (a_p^\perp \vee (a_p^\perp)_p^\perp)_p^\perp = (a_p^\perp \vee a)_p^\perp = (a \vee a_p^\perp)_p^\perp = 1_p^\perp = 1 \bullet p = p.$$

Hence, we have shown that $a_p^\perp$ is an orthocomplement of $a \in [p, 1]$ in the interval $[p, 1]$. □

Corollary. *Let $\mathcal{A} = (A; \bullet, 1)$ be an implication orthoalgebra. Then $\mathcal{A}$ is a set-theoretical union of strong ortholattices with the common greatest element 1 where the lattice operations coincide on the overlapping parts.*

In what follows, we give a certain description of congruences on implication orthoalgebras. Consider a congruence Θ on an implication orthoalgebra $\mathcal{A} = (A; \bullet, 1)$. The class $[1]_\Theta$ will be called the *kernel* of Θ. Hence, each $\Theta \in Con(\mathcal{A})$ determines its kernel. However, also vice versa, each congruence on $\mathcal{A}$ is uniquely determined by its kernel:

Theorem 3. *Let $\mathcal{A} = (A; \bullet, 1)$ be an implication orthoalgebra and $\Theta, \Phi \in Con(\mathcal{A})$. If $[1]_\Theta = [1]_\Phi$ then $\Theta = \Phi$.*

Proof: Assume $[1]_\Theta = [1]_\Phi$ for $\Theta, \Phi \in Con(\mathcal{A})$ and let $(a, b) \in \Theta$. Then clearly

$$\langle a \bullet b, 1 \rangle = \langle a \bullet b, a \bullet a \rangle \in \Theta$$

$$\langle b \bullet a, 1\rangle = \langle b \bullet a, b \bullet b\rangle \in \Theta$$

thus $a \bullet b, b \bullet a \in [1]_\Theta = [1]_\Phi$ and hence

$$\langle a \bullet b, 1\rangle \in \Phi \quad \text{and} \quad \langle b \bullet a, 1\rangle \in \Phi.$$

Using Theorem 1(a),(b) we obtain

$$\langle (a \bullet b) \bullet b, b\rangle = \langle (a \bullet b) \bullet b, 1 \bullet b\rangle \in \Phi,$$

$$\langle (b \bullet a) \bullet a, a\rangle = \langle (b \bullet a) \bullet a, 1 \bullet a\rangle \in \Phi.$$

Hence

$$b\, \Phi\, (a \bullet b) \bullet b = (b \bullet a) \bullet a\, \Phi\, a$$

giving $(a, b) \in \Phi$, i.e. $\Theta \subseteq \Phi$. Analogously we can show $\Phi \subseteq \Theta$, thus $\Theta = \Phi$. □

To describe a congruence Θ on an implication orthoalgebra $\mathcal{A}$, it is enough to characterize its kernel $[1]_\Theta$.

Theorem 4. *Let $\mathcal{A} = (A; \bullet, 1)$ be an implication orthoalgebra and $D \subseteq A$ such that $1 \in D$. The following conditions are equivalent:*

(1) *D is a kernel of some $\Theta \in Con(\mathcal{A})$;*
(2) *D satisfies the following conditions:*

(D1) *if $x \in D$ and $y \bullet z \in D$ then $(x \bullet y) \bullet z \in D$*
(D2) *if $x \bullet y \in D$ and $y \bullet x \in D$ then*

$$(x \bullet z) \bullet (y \bullet z) \in D \quad \textit{and} \quad (z \bullet x) \bullet (z \bullet y) \in D.$$

Proof: It is an easy exercise to verify that every congruence kernel satisfies the conditions (D1) and (D2).

Conversely, let $1 \in D \subseteq A$ and D satisfy (D1) and (D2). Introduce a binary relation Θ_D on A as follows:

(A) $(x, y) \in \Theta_D$ iff $x \bullet y$ and $y \bullet x \in D$.
Evidently, Θ_D is reflexive and symmetric. Suppose $(x, y) \in \Theta_D$ and $(y, z) \in \Theta_D$. Then $x \bullet y, y \bullet x, y \bullet z, z \bullet y \in D$ and, applying (c) of Theorem 1, we obtain

(B) $((x \vee y) \bullet z) \bullet (x \bullet z) = (((x \bullet y) \bullet y) \bullet z) \bullet (x \bullet z) = 1 \in D$.
Further, $x \bullet y \in D$ and $y \bullet z \in D$ imply by (D1)

(C) $((x \bullet y) \bullet y) \bullet z \in D$.
Applying (D1) once more for $x = y$, we derive

$$x \in D \text{ and } x \bullet z \in D \text{ imply } z = (x \bullet x) \bullet z \in D.$$

We use the above rule together with (B) and (C) to obtain $x \bullet z \in D$. Analogously, one can show $z \bullet y \in D$, i.e. $(x, z) \in \Theta_D$ and Θ_D is also transitive. It is an easy calculation to show that (D2) together with the transitivity of Θ_D imply the substitution property with respect to $\bullet$, i.e. Θ_D is a congruence on $\mathcal{A}$.

It follows directly by (A) that D is the kernel of Θ_D. □

In what follows, we are going to characterize congruence kernels as the so-called ideals. Let $\mathcal{A} = (A; \bullet, 1)$ be an implication orthoalgebra. A subset $I \subseteq A$ is called an *ideal* of $\mathcal{A}$ whenever there exists a congruence Θ on $\mathcal{A}$ such that I is the kernel of Θ. It is clear that each congruence Θ determines its kernel $[1]_\Theta$. However, also the converse statement is true by Theorem 3.

This result motivates us to describe ideals of implication orthoalgebras since every ideal determines just one congruence and every congruence is determined by an ideal.

For this, introduce the following concept adapted from Ursini (1972): a term $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is called an *ideal term* of $\mathcal{A} = (A; \bullet, 1)$ in $y_1, \ldots, y_m$ whenever $t(x_1, \ldots, x_n, 1, \ldots, 1) = 1$ is an identity in $\mathcal{A}$.

Lemma 1. *Let $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be an ideal term in $y_1, \ldots, y_m$ of an implication orthoalgebra $\mathcal{A} = (A; \bullet, 1)$ and I be an ideal of $\mathcal{A}$. If $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in I$ then $t(a_1, \ldots, a_n, b_1, \ldots, b_m) \in I$.*

Proof: Let I be an ideal of $\mathcal{A}$. Then there exists a congruence Θ on $\mathcal{A}$ with $I = [1]_\Theta$. Assume further $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in I$. Then $\langle b_i, 1\rangle \in \Theta$ for $i = 1, \ldots, m$ and hence

$$\begin{aligned} &\langle t(a_1, \ldots, a_n, b_1, \ldots, b_m), 1\rangle \\ &= \langle t(a_1, \ldots, a_n, b_1, \ldots, b_m), t(a_1, \ldots, a_n, 1, \ldots, 1)\rangle \in \Theta \end{aligned}$$

thus $t(a_1, \ldots, a_n, b_1, \ldots, b_m) \in [1]_\Theta = I$. □

In other words, every ideal I of $\mathcal{A}$ is *closed under each ideal term* of $\mathcal{A}$. Our goal is to show the crucial result, namely to prove that I is an ideal of $\mathcal{A}$ iff I is closed with respect to a *finite* number of ideal terms which will be explicitly exhibited. Since every congruence kernel is closed with respect to substitutions (D1), (D2) as shown in Theorem 4, we need only to set up these terms and to verify that I satisfies (D1), (D2) whenever it is closed with respect to them (the converse follows by Lemma 1).

Lemma 2. *Let I be a non-void subset of an implication orthoalgebra $\mathcal{A}$ closed under the following ideal terms of $\mathcal{A}$:*

$$t_1(x, y) = x \bullet y$$

$$t_2(x_1, x_2, y_1, y_2) = (x_1 \bullet x_2) \bullet [y_2 \bullet ((y_1 \bullet x_1) \bullet x_2)]$$

$$t_6(x, y_1, y_2) = (y_1 \bullet (y_2 \bullet x)) \bullet x.$$

Then I satisfies the implication (D1).

Proof: At first we we show that I satisfies the property

(1) $a \in I$ and $a \bullet b \in I \;\Rightarrow\; b \in I$.
Indeed, putting $y_1 := a \bullet b$, $y_2 := a$, $x := b$ in the term t_6 we get

$$t_6(b, a \bullet b, a) = [(a \bullet b) \bullet (a \bullet b)] \bullet b = 1 \bullet b = b \in I$$

and (1) is proved.
Assume $x, y \bullet z \in I$. Since $x \in I$, the closedness of I under t_1 gives us

(2) $t_1(y \bullet x, x) = (y \bullet x) \bullet x = (x \bullet y) \bullet y \in I$.
Analogously, taking $x_1 := y$, $x_2 := z$, $y_1 := x$, $y_2 := (x \bullet y) \bullet y$ in t_2 we obtain

(3) $t_2(y, z, x, (x \bullet y) \bullet y) = (y \bullet z) \bullet [((x \bullet y) \bullet y) \bullet ((x \bullet y) \bullet z)] \in I$.
Further, $y \bullet z \in I$, hence applying (1) for $a := y \bullet z$ and $b := ((x \bullet y) \bullet y) \bullet ((x \bullet y) \bullet z)$, we get

$$((x \bullet y) \bullet y) \bullet ((x \bullet y) \bullet z) \in I.$$

Finally, using (1) again for $a := (x \bullet y) \bullet y$ and $b := (x \bullet y) \bullet z$ gives us $(x \bullet y) \bullet z \in I$, finishing the proof. □

To guarantee the closedness of a given subset I under the remaining property (D2), we need the following two lemmas:

Lemma 3. *Let I be a non-void subset of an implication orthoalgebra $\mathcal{A}$ closed under the ideal terms t_6 and*

$$t_3(x_1, x_2, y) = (x_1 \bullet x_2) \bullet (x_1 \bullet (y \bullet x_2));$$

$$t_4(x_1, x_2, x_3, y) = [(x_1 \bullet x_2) \bullet (x_1 \bullet (y \bullet x_3))] \bullet ((x_1 \bullet x_2) \bullet (x_1 \bullet x_3)).$$

Then I has the property

$$x \bullet y \in I \text{ and } y \bullet x \in I \;\Rightarrow\; (z \bullet x) \bullet (z \bullet y) \in I.$$

Proof: Assume $x \bullet y, y \bullet x \in I$ for some $x, y \in A$. Using t_3 for $x_1 := z$, $x_2 := x$, $y := y \bullet x$ we obtain

(4) $t_3(z, x, y \bullet x) = (z \bullet x) \bullet [z \bullet ((y \bullet x) \bullet x)] \in I$.
Substituting $x_1 := z$, $x_2 := x$, $x_3 := y$, $y := x \bullet y$ in t_4, we obtain

(5) $t_4(z, x, y, x \bullet y) = [(z \bullet x) \bullet (z \bullet ((x \bullet y) \bullet y)] \bullet ((z \bullet x) \bullet (z \bullet y)) \in I$.

The closedness of I under t_6 guarantees by Lemma 2 that (1) holds for I, hence (4), (5) and (b) of Theorem 1 yield

$$(z \bullet x) \bullet (z \bullet y) \in I,$$

and we are done. □

Lemma 4. *Let I be a non-void subset of an implication orthoalgebra $\mathcal{A}$ closed under the ideal terms t_2, t_6 and $t_5(x_1, x_2, x_3, y) = [(x_1 \bullet x_2) \bullet ((y \bullet x_3) \bullet x_2)] \bullet ((x_1 \bullet x_2) \bullet (x_3 \bullet x_2))$.*
Then I has the property

$$x \bullet y \in I \text{ and } y \bullet x \in I \Rightarrow (x \bullet z) \bullet (y \bullet z) \in I.$$

Proof: The closedness of I under t_2 immediately yields by putting $y_2 := 1$ also the closedness under

$$t'(x_1, x_2, y) = (x_1 \bullet x_2) \bullet ((y \bullet x_1) \bullet x_2).$$

Let us substitute $x_1 := x, x_2 := z, y := y \bullet x$ in t'. This gives us

$$(x \bullet z) \bullet (((y \bullet x) \bullet x) \bullet z) \in I.$$

Moreover, $(y \bullet x) \bullet x = (x \bullet y) \bullet y$, hence also

(6) $(x \bullet z) \bullet (((x \bullet y) \bullet y) \bullet z) \in I.$

Now, considering t_5 for instances $x_1 := x, x_2 := z, x_3 := y, y := x \bullet y$, we have

(7) $[(x \bullet z) \bullet (((x \bullet y) \bullet y) \bullet z)] \bullet ((x \bullet z) \bullet (y \bullet z)) \in I.$

The closedness of I under t_6 gives us by Lemma 2 that I satisfies the property (1). This together with (6) and (7) leads to

$$(x \bullet z) \bullet (y \bullet z) \in I.$$

□

Applying the previous lemmas, we obtain the desired description of ideals in implication orthoalgebras:

Theorem 5. *Let I be a non-void subset of an implication orthoalgebra $\mathcal{A}$. Then I is an ideal of $\mathcal{A}$ iff I is closed with respect to the ideal terms $t_1, t_2, t_3, t_4, t_5, t_6$.*

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